

CURVED FLATS, PLURIHARMONIC MAPS AND CONSTANT CURVATURE IMMERSIONS INTO PSEUDO-RIEMANNIAN SPACE FORMS

DAVID BRANDER

ABSTRACT. We study two aspects of the loop group formulation for isometric immersions with flat normal bundle of space forms. The first aspect is to examine the loop group maps along different ranges of the loop parameter. This leads to various equivalences between global isometric immersion problems among different space forms and pseudo-Riemannian space forms. As a corollary, we obtain a non-immersibility theorem for spheres into certain pseudo-Riemannian spheres and hyperbolic spaces.

The second aspect pursued is to clarify the relationship between the loop group formulation of isometric immersions of space forms and that of pluriharmonic maps into symmetric spaces. We show that the objects in the first class are, in the real analytic case, extended pluriharmonic maps into certain symmetric spaces which satisfy an extra reality condition along a totally real submanifold. We show how to construct such pluriharmonic maps for general symmetric spaces from curved flats, using a generalised DPW method.

1. INTRODUCTION

It is well known that harmonic maps from a Riemann surface into a symmetric space are integrable systems which can be approached successfully using loop group techniques, which followed from the work of Uhlenbeck [20] on harmonic maps from S^2 into a Lie group. Further, various geometrical problems, such as constant mean curvature surfaces, have been studied successfully by showing that they are such harmonic maps (see, for example, [9] for an introduction). In higher dimensions, pluriharmonic maps into symmetric spaces were also shown to have a similar approach by Ohnita and Valli [14], although the applications to special submanifolds appear to be little explored thus far. It turns out, as we will show, that the loop group maps corresponding to isometric immersions of space forms, which were defined by Ferus and Pedit [8], are a special case of such pluriharmonic maps.

1.1. Isometric immersions of space forms. In this paper, we first study, in Section 3, the interpretations of these loop group maps for different ranges of the spectral parameter. We show, in Theorem 3.2, that the map corresponding to each isometric immersion problem actually contains three families of immersions, into different space forms and pseudo-Riemannian space forms, for values of the parameter along \mathbf{R}^* , $i\mathbf{R}^*$ or S^1 , denoting the non-zero real, imaginary and unitary numbers respectively. It is also observed (Remark 3.3) that constant curvature immersions with flat normal bundle into other pseudo-Riemannian space forms, beyond those arising here, have an analogous loop group formulation, and solutions can be generated via the AKS theory, as in [8], or by the well known dressing procedure.

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Constant curvature immersions with flat normal bundle into pseudo-Riemannian space forms have previously been studied by Barbosa, Ferreira and Tenenblat [1], and Dafeng, Qing and Yi [4], with some additional assumptions to ensure the existence of special coordinates. The loop group formulation given here has what is perhaps an advantage, in that it is coordinate-free, and therefore applies to all cases.

This formulation is global, and we also prove, in Proposition 3.1, that, for a given loop group map, completeness is equivalent among all the isometric immersions obtained from it.

A corollary of our results is the equivalence among various global isometric immersion problems of space forms into pseudo-Riemannian space forms, stated in Corollary 3.6. Among the applications of this is a coordinate-free proof, Corollary 3.7, of the known result that the problems of globally isometrically immersing the hyperbolic space H^m into a space form Q_c^{2m-1} , for $-1 < \tilde{c}$, with $\tilde{c} \neq 0$, are equivalent, that is, independent of \tilde{c} . This is related to the conjecture that no such isometric immersion is possible for any $\tilde{c} > -1$, an extension of Hilbert's result regarding immersions of the hyperbolic plane into Euclidean 3-space. Following work by Pedit and Xavier, [15], [22], this conjecture has been proven under the assumption that the fundamental group of the manifold being immersed is non-trivial, by Nikolayevsky [13], but remains a conjecture for the simply connected case. One can alternatively show that the cases are equivalent for all $\tilde{c} > -1$ by using JD Moore's global principle coordinates [12], and reducing the problem to finding a global solution for the generalised sine-Gordon equation, studied by Terng and Tenenblat in [17], [18].

Another application, Corollary 3.8, is that there is no global isometric immersion with flat normal bundle of a sphere $S^m(R)$, of radius $R < 1$, into the pseudo-Riemannian sphere S_{m-1}^{2m-1} , or of a sphere of any radius into the pseudo-Riemannian hyperbolic space H_{m-1}^{2m-1} .

1.2. Relation to pluriharmonic maps. The remainder of the paper, beginning in Section 4, is devoted to exploring the relations between the loop group formulation for isometric immersions of space forms, and that of pluriharmonic maps into a symmetric space.

We show, in Theorem 7.1, how to construct special pluriharmonic maps whose extended families satisfy an extra reality condition when restricted to a certain totally real submanifold. They are constructed from real analytic curved flats into a different symmetric space. Many examples of such curved flats can be constructed via the AKS theory [7].

As a special case, we show, in Theorem 8.1, that the loop group maps for isometric immersions studied in Section 3 are, in the real analytic case, just restrictions to a totally real submanifold of pluriharmonic maps into certain symmetric spaces.

The proof given here uses the generalised DPW method [6], [2], which associates a certain type of loop group map, which includes both the isometric immersions of space forms and pluriharmonic maps, to a simpler map, essentially a curved flat. This is then extended to a holomorphic map from a complex manifold into the loop group associated to the complexification of the Lie group. We then apply the DPW correspondence again, and obtain a pluriharmonic map. In general the DPW method can only be applied to elements which are in the big cell of the loop group, but we are able to get around this problem in this case by renormalizing at different points of M .

This example suggests that pluriharmonic maps into symmetric spaces, perhaps satisfying extra conditions such as an additional reality condition, should yield solutions to other interesting problems in geometry.

In Section 8.2 we recharacterize the problem of identifying which totally geodesic submanifolds with flat normal bundle of the sphere and hyperbolic space (which is a degenerate

case in the formulation of Ferus and Pedit) belong to the families defined in [8], in terms of pluriharmonic maps.

2. LIMITED CONNECTION ORDER MAPS INTO LOOP GROUPS

In this section we outline some definitions and terminology. For further details, we refer the reader to [2]. Let G be a complex Lie group, with Lie algebra \mathfrak{g} . Let ΛG be the group of real analytic maps from the unit circle S^1 into G , with a topology that makes ΛG a Banach Lie group. Any element γ of ΛG has an extension to a holomorphic map into G on some annulus, \mathcal{A}_γ , containing S^1 , and in fact all the examples considered here are holomorphic on $\mathbf{C} \setminus \{0\} := \mathbf{C}^*$. Let M be a smooth manifold and denote by $\Lambda G(M)$ the group of smooth maps $M \rightarrow \Lambda G$ normalised to the identity at some fixed base point $p \in M$. If $F \in \Lambda G(M)$, then for each value of the loop parameter $\lambda \in \mathcal{A}_F$, we can expand the Maurer-Cartan form $F_\lambda^{-1}dF_\lambda$ as a Fourier series in λ ,

$$F_\lambda^{-1}dF_\lambda = \sum_i A_i \lambda^i.$$

For any subgroup \mathcal{H} of the loop group ΛG , and any extended integers $a \leq b$, we define $\mathcal{H}(M)_a^b$ to be the set of elements in $\mathcal{H}(M)$ whose Maurer-Cartan form is of bottom and top degree a and b respectively in λ , and call elements of these sets *connection order $\binom{b}{a}$ maps*.

Let \mathcal{H}^0 denote the subgroup of constant loops in \mathcal{H} . Note that \mathcal{H}^0 is a subgroup of $G = \Lambda G^0$. The natural objects of study in this paper are maps into the quotient space $\frac{\mathcal{H}}{\mathcal{H}^0}$ (where the action is right multiplication), and we denote the set of connection order $\binom{b}{a}$ maps from M into this group by $\frac{\mathcal{H}}{\mathcal{H}^0}(M)_a^b$. For a fixed value of λ these are maps from M into some quotient space $\frac{U}{\mathcal{H}^0}$, where U is some subgroup of G .

2.1. Some further terminology. Let $\Lambda^\pm G$ denote the subgroups of ΛG consisting of loops which extend analytically to D^\pm , where D^+ is the unit disc and D^- is the complement of its closure in the Riemann sphere. $\Lambda_e^+ G$ is the subgroup of $\Lambda^+ G$ whose elements are normalised to the identity at $\lambda = 0$, and $\Lambda_e^- G$ is the analogue at $\lambda = \infty$. If \mathcal{H} is a subgroup of ΛG then the intersection of \mathcal{H} with these subgroups are accordingly denoted \mathcal{H}^\pm and \mathcal{H}_e^\pm . The loop group ΛG is called *Birkhoff decomposable* if, on an open dense neighbourhood of the identity, called the big cell BAG , there is a (left Birkhoff) decomposition

$$(1) \quad \Lambda G = (\Lambda_e^+ G) \cdot (\Lambda^- G),$$

and the map from $\Lambda_e^+ G \times \Lambda^- G$ to this open set is an analytic diffeomorphism. There is also an analogous right Birkhoff decomposition, substituting \pm for \mp . In this paper, the group G is always complex semisimple, so ΛG is Birkhoff decomposable [16]. A subgroup \mathcal{H} of ΛG is called Birkhoff decomposable if, in the decomposition given by (1), both factors on the right hand side are also in the subgroup.

Let ϕ be an automorphism of a subgroup \mathcal{H} , which is an extension of an automorphism of \mathcal{H}^0 . We will say that ϕ is *positive* or *negative (holomorphic)* if it takes \mathcal{H}^\pm to \mathcal{H}^\pm or \mathcal{H}^\pm to \mathcal{H}^\mp respectively. Note that a negative automorphism must be of even order. A commonly used positive automorphism is the twisting:

$$(2) \quad (\sigma X)(\lambda) := \sigma^0 X(-\lambda),$$

where σ^0 is an involution of the Lie group G .

A positive or negative holomorphic *reality condition* on \mathcal{H} is a positive or negative holomorphic extension to \mathcal{H} of a reality condition on \mathcal{H}^0 . Examples we will use are: for ρ^0 and

τ^0 reality conditions on G , define positive ρ and negative τ on ΛG by

$$(3) \quad \rho x(\lambda) := \rho^0 x(\bar{\lambda}),$$

$$(4) \quad \tau x(\lambda) := \tau^0 x(1/\bar{\lambda}).$$

Note that x is understood to be defined on some annulus \mathcal{A}_x around S^1 , and so $x(1/\bar{\lambda}) \neq x(\lambda)$. For any automorphism ϕ of ΛG denote its fixed point subgroup by ΛG_ϕ . Elements of ΛG_ρ are real (that is take values in the real form of G determined by ρ^0) for real values of $\lambda \in \mathcal{A}$, while elements of ΛG_τ are real for $\lambda \in S^1$. For every loop group automorphism used in this paper, the corresponding automorphism of the Lie algebra is given by the same formula, so we will use the same notation to denote it.

The important fact to note is that for any Birkhoff decomposable subgroup \mathcal{H} and any *positive* holomorphic finite order automorphism ϕ , the fixed point subgroup \mathcal{H}_ϕ is also Birkhoff decomposable. The same does not hold if ϕ is negative, as, given a Birkhoff decomposition $x = x_+ y_-$ according to (1), then x_+ cannot be fixed by ϕ in general.

3. CONSTANT CURVATURE IMMERSIONS INTO PSEUDO-RIEMANNIAN SPACE FORMS

In this section, we first sketch the loop group construction of Ferus and Pedit [8] for isometric immersions of space forms. We then evaluate these loop group maps along other ranges of the spectral parameter, to obtain several equivalences between isometric immersions into space forms and pseudo-Riemannian space forms.

3.1. The loop group formulation for isometric immersions of space forms. Here is a brief outline of the formulation from [8]. In that work it was also shown how to construct many examples of these maps using the AKS theory. Let M be a simply connected manifold of dimension m . We first consider the case that the target space is a sphere. The loop group maps are elements of

$$(5) \quad \frac{\mathcal{H}_{\rho\mu}(M)^1}{\mathcal{H}_{\rho\mu}^0}{}_{-1},$$

where $\mathcal{H} = \Lambda G_\sigma$, $G = SO(m+k+1, \mathbf{C})$, σ is given by the equation (2) for the involution σ^0 defining a symmetric space $SO(m+k+1)/(SO(m) \times SO(k+1))$, namely

$$\sigma^0 := \text{Ad}_P,$$

for $P = \text{diag}(I_m, -I_{k+1})$, and μ is the negative involution

$$(6) \quad (\mu X)(\lambda) := \text{Ad}_Q(X(1/\lambda)),$$

for $Q = \text{diag}(I_{m+1}, -I_k)$. Here I_j is the $j \times j$ identity matrix. Finally, ρ is one of three reality conditions, described below, and $\mathcal{H}_{\rho\mu}$ is the subgroup of \mathcal{H} fixed by both involutions ρ and μ .

There are essentially three cases for the induced Gauss curvature on the immersion, which is constant for a fixed value of λ , but varies with λ . These correspond to three different choices for the reality condition ρ , and are displayed in Table 1.

Reality condition	Parameter range	Induced Gauss curvature
$(\rho_1 X)(\lambda) := \overline{X(-\lambda)}$	$\lambda \in i\mathbf{R}^*$	$c_\lambda \in (-\infty, 0)$
$(\rho_2 X)(\lambda) := \overline{X(\bar{\lambda})}$	$\lambda \in \mathbf{R}^*$	$c_\lambda \in (0, 1]$
$(\rho_3 X)(\lambda) := \overline{X(1/\bar{\lambda})}$	$\lambda \in S^1$	$c_\lambda \in [1, \infty)$

TABLE 1. Cases 1-3 for immersions into a sphere

If U is a subset of M , an *adapted frame* for an immersion $f : U \rightarrow S^{m+k}$ is defined to be a map $F = [e_1, \dots, e_m, f, \xi_1, \dots, \xi_k] : U \rightarrow SO(m+k+1)$, whose first m and last k columns span the tangent and normal bundles to the image $f(U)$. The involutions σ and μ mean that if F is any representative of an element of $\frac{\mathcal{H}_{\rho\mu}}{\mathcal{H}_{\rho\mu}^0}(M)_{-1}^1$, then, for a fixed value of λ , F has the interpretation as an adapted frame for a map $f : M \rightarrow S^{m+k}$, provided the $(m+1)$ 'th column, f , is an immersion. The map f is independent of the choice of representative F , because \mathcal{H}_{μ}^0 is just the subgroup $SO(m) \times I \times SO(k)$, which acts on the right by fixing f and changing the orthonormal frames for the tangent and normal spaces. The involution μ also ensures that the derivative df has no component in the directions of any of ξ_1, \dots, ξ_k .

Notationally, we will not normally distinguish such a representative F from its equivalence class, and we will call either an *extended frame* for the family of immersions f^λ . An element F of $\mathcal{H}_{\rho\mu}(M)_{-1}^1$ has a Maurer-Cartan form which looks like:

$$(7) \quad F^{-1}dF = \begin{bmatrix} \omega & & \\ -[\epsilon(\lambda + \lambda^{-1})\theta & (\lambda - \lambda^{-1})\beta]^t & \begin{bmatrix} (\lambda + \lambda^{-1})\theta & (\lambda - \lambda^{-1})\beta \\ \eta \end{bmatrix} \end{bmatrix},$$

where the first row and column of η are zero, and, in the spherical case, $\epsilon = 1$. The 1-forms ω and η are the connections of the tangent and normal bundles respectively, $(\lambda + \lambda^{-1})\theta$ is the dual frame to our tangent frame, and $(\lambda - \lambda^{-1})\beta$ is the second fundamental form.

A straightforward computation shows that $F^{-1}dF$ satisfying the Maurer-Cartan equation for all λ is equivalent to the integrability of F at a single value of λ plus the extra conditions

$$(8) \quad d\omega + \omega \wedge \omega = 4\theta \wedge \theta^t,$$

$$(9) \quad d\eta + \eta \wedge \eta = 0.$$

The second equation says that the normal bundle is flat, and equation (8) says that the induced sectional curvature on the image of f is

$$c_\lambda = \frac{4}{(\lambda + \lambda^{-1})^2},$$

which follows from the fact that the coframe is $(\lambda + \lambda^{-1})\theta$. One then checks that as λ varies over the ranges $i\mathbf{R}^*$, \mathbf{R}^* , and S^1 , c_λ varies over the intervals in Table 1. The loop group map F is well defined on \mathbf{C}^* , but the map f is not an immersion at $\lambda = \pm i$, since the coframe necessarily vanishes there.

If we allow our immersions to be degenerate at some points, meaning the derivative drops rank, then any element of $\frac{\mathcal{H}_{\rho\mu}}{\mathcal{H}_{\rho\mu}^0}(M)_{-1}^1$, corresponds to a family of isometric immersions.

Conversely, if M is simply connected, then, once we fix the base point $p \in M$ at which our elements of $\mathcal{H}_{\rho\mu}(M)$ are normalised, there is a unique element of $\frac{\mathcal{H}_{\rho\mu}}{\mathcal{H}_{\rho\mu}^0}(M)_{-1}^1$ associated to a given isometric immersion with constant curvature in the appropriate range of the table, with the exception of the limiting value $c_\lambda = 1$ in Cases 2 and 3. This corresponds to the values $\lambda = \pm 1$, at which point the second fundamental form, $(\lambda - \lambda^{-1})\beta$, vanishes, so the immersion is totally geodesic. However, given a totally geodesic immersion, we cannot insert λ into its Maurer-Cartan form to obtain the family, as we do not know what β should be.

This converse statement was shown locally in [8], by choosing an adapted frame for the immersion. A single global adapted frame may not exist, but the equivalence class in the space $\frac{\mathcal{H}_{\rho\mu}}{\mathcal{H}_{\rho\mu}^0}(M)_{-1}^1$ is nevertheless well defined:

Lemma 3.1. *Let M be a simply connected manifold of dimension m , and $f : M \rightarrow S^{m+k}$ an immersion with flat normal bundle and induced constant curvature c , with $0 \neq c \neq 1$, with the normalisation $f(p) = [0, \dots, 1, 0, \dots, 0]$, the standard unit vector E_{m+1} . Then there*

is a unique element $F \in \frac{\mathcal{H}_{\rho\mu}}{\mathcal{H}_{\rho\mu}^0}(M)_{-1}^1$, where ρ is the appropriate reality condition, whose $(m+1)$ 'th column, evaluated at $\lambda_0 = \frac{1}{\sqrt{c}}(1 + \sqrt{1-c})$, is f .

Proof. For any point q of M , there is a simply connected neighbourhood, U_q , of q , which contains p , and an adapted frame, $F_q = [e_1, \dots, e_m, f, \xi_1, \dots, \xi_k]$, on U_q , normalised to the identity at p . This can be obtained by parallel translating the identity matrix along some path from p to q , and then extending to a neighbourhood of that path. The Maurer-Cartan form of F_q is

$$A_q := F_q^{-1}dF_q = \begin{bmatrix} & \omega & \\ -[\theta & \beta]^t & \begin{bmatrix} \theta & \beta \\ \eta \end{bmatrix} \end{bmatrix}.$$

Following [8], to insert the parameter λ , one multiplies θ by $\frac{\sqrt{c}}{2}(\lambda + \lambda^{-1})$ and β by $\frac{\sqrt{c}}{2\sqrt{1-c}}(\lambda - \lambda^{-1})$, and then integrates on U_q , with the initial condition $F(p) = I$, to get a representative $F_q^\lambda \in \mathcal{H}_{\rho\mu}(U_q)_{-1}^1$.

We need to check that for any other point r , F_q^λ and F_r^λ differ only by post-multiplication by maps into $\mathcal{H}_{\rho\mu}^0$, on the intersections of their domains of definition. Now at $x \in U_q \cap U_r$, we have $F_r(x) = F_q(x)G(x)$ for some G which is smooth and takes values in $\mathcal{H}_{\rho\mu}^0 = SO(m) \times 1 \times SO(k)$, because F_q and F_r are both adapted frames. The matrix G is of the form $\text{diag}(A, 1, B)$, and hence

$$F_r^{-1}dF_r = \begin{bmatrix} A^t\omega A + A^tdA & \begin{bmatrix} A^t\theta & A^t\beta B \end{bmatrix} \\ -[A^t\theta & A^t\beta B]^t & B^t\eta B + B^tdB \end{bmatrix}.$$

It follows from the construction of the extended frame F_r^λ , that it has the same Maurer-Cartan form at x as $F_q^\lambda G$. Since both functions are equal at x , and their Maurer-Cartan forms agree, it follows that F_r^λ and $F_q^\lambda G$ agree wherever they are defined. \square

The case where the target is hyperbolic space has the same formulation, replacing the group $SO(m+k+1, \mathbf{C})$ with $SO(m+k, 1, \mathbf{C})$, defined here to be the subgroup of $GL(m+k+1, \mathbf{C})$ consisting of matrices which preserve the bilinear form given by the matrix

$$J := \text{diag}(I_m, -1, I_k).$$

In this case, $\epsilon = -1$ in (7), and the corresponding induced curvatures in Table 1 are all negated.

We will only need one of the cases for the hyperbolic space and we therefore add to Cases 1-3 above, the following:

Case 4: $\Lambda SO(m+k, 1, \mathbf{C})_{\sigma\mu\rho_2}(M)_{-1}^1$, where σ, μ and ρ_2 are as before. Elements of $\Lambda SO(m+k, 1, \mathbf{C})_{\sigma\mu\rho_2}(M)_{-1}^1$ are isometric immersions with flat normal bundle $M \rightarrow H^{m+k+1}$ with induced constant curvature c_λ in the interval $[-1, 0)$.

3.2. Interpretation for other ranges of the spectral parameter. The goal of this subsection is to identify for each of Cases 1-4 above, the different maps obtained for values of the spectral parameter in all three ranges \mathbf{R} , $i\mathbf{R}$ and S^1 . This was partially investigated in [2], where the last row of Table 2 below and the first row of Table 3 were found.

Let \mathbf{R}_s^n denote the *pseudo-Euclidean* space \mathbf{R}^n equiped with a metric \langle, \rangle with signature $(s, n-s)$, that is, it is isometric to a space with metric $J = \text{diag}(-I_s, I_{n-s})$. Define the pseudo-Riemannian sphere, S_s^n , and pseudo-Riemannian hyperbolic space, H_s^n , by

$$\begin{aligned} S_s^n &:= \{x \in \mathbf{R}_s^{n+1} : \langle x, x \rangle = 1\}, \\ H_s^n &:= \{x \in \mathbf{R}_{s+1}^{n+1} : \langle x, x \rangle = -1\}. \end{aligned}$$

These two spaces are complete pseudo-Riemannian manifolds, both with signature $(s, n-s)$, and with constant sectional curvatures 1 and -1 respectively (see, for example, [21]).

Theorem 3.2. *Let G and H denote the groups $SO(m+k+1, \mathbf{C})$ and $SO(m+k, 1, \mathbf{C})$ respectively. Let F be an element of either $\frac{\Lambda G_{\sigma\mu\rho i}}{\Lambda G_{\sigma\mu\rho i}^0}(M)_{-1}^1$, $i = 1, \dots, 3$, or $\frac{\Lambda H_{\sigma\mu\rho 2}}{\Lambda H_{\sigma\mu\rho 2}^0}(M)_{-1}^1$, as described for Cases 1-4 above. If the $(m+1)$ 'th column, f , of F is an immersion for all $\lambda \in \mathbf{C}^* \setminus \{\pm i\}$, then, by evaluating f for values of λ in the ranges indicated in Tables 2-5 below, isometric immersions with flat normal bundle into the pseudo-Riemannian space forms displayed are obtained. In all cases, the induced metric on M is positive definite and with constant sectional curvature varying through the range indicated.*

Conversely, if M is simply connected, then any isometric immersion with flat normal bundle into one of the target spaces displayed, with constant sectional curvature in the corresponding range, apart from the cases $c = 1$ when the target space has positive curvature, and $c = -1$ when the target space has negative curvature, belongs to one of these families.

Parameter range	Induced Gauss curvature	Target space
$\lambda \in i\mathbf{R}^* \setminus \{\pm i\}$	$c_\lambda \in (-\infty, 0)$	S^{m+k}
$\lambda \in \mathbf{R}^*$	$c_\lambda \in [-1, 0)$	H_k^{m+k}
$\lambda \in S^1 \setminus \{\pm i\}$	$c_\lambda \in (-\infty, -1]$	H^{m+k}

TABLE 2. Case 1: $\frac{\Lambda G_{\sigma\mu\rho 1}}{G_{\sigma\mu\rho 1}^0}(M)_{-1}^1$

Parameter range	Induced Gauss curvature	Target space
$\lambda \in i\mathbf{R}^* \setminus \{\pm i\}$	$c_\lambda \in (0, \infty)$	H_k^{m+k}
$\lambda \in \mathbf{R}^*$	$c_\lambda \in (0, 1]$	S^{m+k}
$\lambda \in S^1 \setminus \{\pm i\}$	$c_\lambda \in [1, \infty)$	S_k^{m+k}

TABLE 3. Case 2: $\frac{\Lambda G_{\sigma\mu\rho 2}}{G_{\sigma\mu\rho 2}^0}(M)_{-1}^1$

Parameter range	Induced Gauss curvature	Target space
$\lambda \in i\mathbf{R}^* \setminus \{\pm i\}$	$c_\lambda \in (0, \infty)$	H^{m+k}
$\lambda \in \mathbf{R}^*$	$c_\lambda \in (0, 1]$	S_k^{m+k}
$\lambda \in S^1 \setminus \{\pm i\}$	$c_\lambda \in [1, \infty)$	S^{m+k}

TABLE 4. Case 3: $\frac{\Lambda G_{\sigma\mu\rho 3}}{G_{\sigma\mu\rho 3}^0}(M)_{-1}^1$

Parameter range	Induced Gauss curvature	Target space
$\lambda \in i\mathbf{R}^* \setminus \{\pm i\}$	$c_\lambda \in (-\infty, 0)$	S_k^{m+k}
$\lambda \in \mathbf{R}^*$	$c_\lambda \in [-1, 0)$	H^{m+k}
$\lambda \in S^1 \setminus \{\pm i\}$	$c_\lambda \in (-\infty, -1]$	H_k^{m+k}

TABLE 5. Case 4: $\frac{\Lambda H_{\sigma\mu\rho 2}}{H_{\sigma\mu\rho 2}^0}(M)_{-1}^1$

Proof. Case 1

Let $F \in \frac{\Lambda G_{\sigma\mu\rho_1}}{G_{\sigma\mu\rho_1}^0}(M)_{-1}^1$. The first row of Table 2 we already know from [8]. To get the second row, we need a reality condition along \mathbf{R} . Consider $\phi := \text{Ad}_T : GL(m+k+1, \mathbf{C}) \rightarrow GL(m+k+1, \mathbf{C})$, where $T := \text{diag}(iI_m, 1, I_k)$. Now ϕ is an isomorphism between $SO(m+k+1, \mathbf{C})$ and $SO(m, k+1, \mathbf{C})$, where the latter group is defined here to be the set of matrices preserving the bilinear form

$$\hat{J} := \text{diag}(I_m, -I_{k+1}).$$

To verify this one checks that $A^t A = I$ is equivalent to $(\phi A)^t \hat{J} \phi = \hat{J}$.

It is also easy to see that ϕ is a bijection between the two sets $\frac{\Lambda G_{\sigma\mu\rho_1}}{G_{\sigma\mu\rho_1}^0}(M)_{-1}^1$ and $\frac{\Lambda SO(m, k+1, \mathbf{C})_{\sigma\mu\rho_2}}{SO(m, k+1, \mathbf{C})_{\sigma\mu\rho_2}^0}(M)_{-1}^1$, since ϕ commutes with both σ and μ and if F satisfies ρ_1 then

$$\begin{aligned} \rho_2(\phi F)(\lambda) &= \overline{TF(\bar{\lambda})T^{-1}} \\ &= \bar{T}P\bar{F}(-\bar{\lambda})P\bar{T}^{-1} \\ &= \bar{T}PF(\lambda)P\bar{T}^{-1} \\ &= (-T)F(\lambda)(-T)^{-1} \\ &= \phi F(\lambda). \end{aligned}$$

To get to the second line we used that fact that $\sigma F(\lambda) = \text{Ad}_P F(-\lambda) = F(\lambda)$.

We now interpret $\hat{F} := \phi(F)$. The analysis of an element $\hat{F} \in \frac{\Lambda SO(m, k+1, \mathbf{C})_{\sigma\mu\rho_2}}{SO(m, k+1, \mathbf{C})_{\sigma\mu\rho_2}^0}(M)_{-1}^1$ is similar to that of the case where the group is $SO(m+k+1, \mathbf{C})$, explained above in Section 3.1: the Maurer-Cartan form has the expression

$$\hat{F}^{-1} d\hat{F} = \begin{bmatrix} \omega & [(\lambda + \lambda^{-1})\theta & (\lambda - \lambda^{-1})\beta] \\ [(\lambda + \lambda^{-1})\theta & (\lambda - \lambda^{-1})\beta] & \eta \end{bmatrix}^t,$$

where the first row and column of η are zeros. This differs from the expression (7) only by a minus sign in the lower left corner. \hat{F} is real for values of λ in \mathbf{R} . For such values of λ , we take the $(m+1)$ 'th column of \hat{F} as our map f , then, by the definition of $SO(m, k+1)$, we have $f^t \hat{J} f = -1$, so f takes values in H_{k+1}^{m+k} . The zeros in the first row and column of η say that the tangent space to the image of f lies in the span of the first m columns, so that if f is an immersion then the first m columns, $\{e_1, \dots, e_m\}$, are a frame for the tangent bundle. Since $\hat{F}^t \hat{J} \hat{F} = \hat{J}$, it follows from the form of \hat{J} that $e_i^t \hat{J} e_j = \delta_{ij}$, so the induced metric is positive definite. As before, the integrability condition implies the equations,

$$(10) \quad d\omega + \omega \wedge \omega = -4\theta \wedge \theta^t,$$

$$(11) \quad d\eta + \eta \wedge \eta = 0.$$

which imply that the normal bundle is flat and the induced Gauss curvature is:

$$c_\lambda = \frac{-4}{(\lambda + \lambda^{-1})^2},$$

which varies over the interval $[-1, 0)$ as λ varies over \mathbf{R}^* . The converse argument is also similar to the spherical case of Lemma 3.1.

The third row of Table 2 was obtained in [2], in a similar fashion, using $T = \text{diag}(iI_m, 1, iI_k)$ and $\hat{J} = \text{diag}(I_m, -1, I_k)$.

Case 2

The second row is given in [8]. To get the first row, proceed as in Case 1, using $T = \text{diag}(iI_m, 1, I_k)$ and $\hat{J} = \text{diag}(I_m, -1, -I_k)$. For the third row, use $T = \text{diag}(iI_m, i, I_k)$ and

$$\hat{J} = \text{diag}(I_m, 1, -I_k).$$

Case 3

The third row is given in [8]. To get the first row, again proceed as in Case 1, using $T = \text{diag}(iI_m, 1, iI_k)$ and $\hat{J} = \text{diag}(I_m, -1, I_k)$. For the second row, use $T = \text{diag}(iI_m, i, I_k)$ and $\hat{J} = \text{diag}(I_m, 1, -I_k)$.

Case 4

The second row we know from [8]. To get the first row, use $T = \text{diag}(I_m, i, iI_k)$ and $\hat{J} = \text{diag}(I_m, 1, -I_k)$. For the third row, use $T = \text{diag}(I_m, 1, iI_k)$ and $\hat{J} = \text{diag}(I_m, -1, -I_k)$. \square

Remark 3.3. It is clear from the preceding proof that isometric immersions with flat normal bundle of a constant curvature Riemannian manifold M^m into either S_l^{m+k} or H_l^{m+k} , for any $0 \leq l \leq k$ can be treated similarly, by starting with the group which preserves the bilinear form $J = \text{diag}[I_m, \epsilon, -I_l, I_{k-l}]$, where $\epsilon = \pm 1$.

Example 3.4. Here is a simple example from Case 3. Consider the family of maps $F_\lambda : \mathbf{R}^2 \rightarrow G = SO(4, \mathbf{C})$ which takes $(u, v) \in \mathbf{R}^2$ to the matrix

$$\begin{bmatrix} \cos(u) & -\sin(u)\sin(v) & a\sin(u)\cos(v) & b\sin(u)\cos(v) \\ 0 & \cos(v) & a\sin(v) & b\sin(v) \\ -a\sin(u) & -a\cos(u)\sin(v) & a^2\cos(u)\cos(v) + b^2 & ab(\cos(u)\cos(v) - 1) \\ -b\sin(u) & -b\cos(u)\sin(v) & ab(\cos(u)\cos(v) - 1) & b^2\cos(u)\cos(v) + a^2 \end{bmatrix},$$

where

$$a = \frac{1}{2}(\lambda + \lambda^{-1}), \quad b = \frac{i}{2}(\lambda - \lambda^{-1}).$$

The Maurer-Cartan form of F_λ is

$$F_\lambda^{-1}dF_\lambda = \begin{bmatrix} 0 & -\sin(v)du & a\cos(v)du & b\cos(v)du \\ \sin(v)du & 0 & a dv & b dv \\ -a\cos(v)du & -a dv & 0 & 0 \\ -b\cos(v)du & -b dv & 0 & 0 \end{bmatrix}.$$

Now $F_\lambda^{-1}dF_\lambda$ is fixed by σ , μ and ρ_3 , so it takes values in the Lie algebra of $\Lambda G_{\sigma\mu\rho_3}$, and $F_\lambda(0,0) = I \in \Lambda G_{\sigma\mu\rho_3}$. Therefore F_λ is a map into $\Lambda G_{\sigma\mu\rho_3}$, and, since its Maurer-Cartan form has top and bottom degree 1 and -1 respectively, it represents an element of $\frac{\Lambda G_{\sigma\mu\rho_3}}{G_{\sigma\mu\rho_3}^0}(\mathbf{R}^2)_{-1}^1$. Thus, according to Table 4, if the third column of F_λ , namely

$$f^\lambda(u, v) = \begin{bmatrix} \frac{1}{2}(\lambda + \lambda^{-1})\sin(u)\cos(v) \\ \frac{1}{2}(\lambda + \lambda^{-1})\sin(v) \\ \frac{1}{4}(\lambda + \lambda^{-1})^2\cos(u)\cos(v) - \frac{1}{4}((\lambda - \lambda^{-1})^2) \\ \frac{i}{4}((\lambda + \lambda^{-1})(\lambda - \lambda^{-1})(\cos(u)\cos(v) - 1)) \end{bmatrix} =: \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix},$$

is an immersion, then, for a value of λ in S^1 , it is an immersion into S^3 with constant Gauss curvature greater or equal to 1. The dual frame for f^λ is given by

$$(12) \quad \theta = \begin{bmatrix} \frac{1}{2}(\lambda + \lambda^{-1})\cos(v)du \\ \frac{1}{2}(\lambda + \lambda^{-1})dv \end{bmatrix},$$

and so, if $\lambda \neq \pm i$, then f^λ is immersive away from the degenerate coordinate lines $\cos(v) = 0$. In fact f^λ is a deformation, through a family of isometrically embedded spheres, of the totally

geodesic embedding of S^2 into S^3 given by

$$f(u, v) = \begin{bmatrix} \sin(u) \cos(v), & \sin(v), & \cos(u) \cos(v), & 0 \end{bmatrix}^t,$$

which is achieved at $\lambda = 1$.

To obtain the isometric immersions from the first two lines of Table 4, we need to apply the transformations Ad_T given in the proof first: using $T = \text{diag}(iI_2, 1, i)$, the third column of $\text{Ad}_T(F_\lambda)$ is

$$\hat{f}^\lambda(u, v) = [if_1, if_2, f_3, if_4]^t,$$

and this is indeed real for values of λ along $i\mathbf{R}$, and gives a family of embeddings of a sphere into H^3 , with constant sectional curvature $c_\lambda = \frac{-4}{(\lambda + \lambda^{-1})^2}$, which varies through the range $(0, \infty)$.

Finally, to get the immersion in the second row of Table 4, we use the matrix $T = \text{diag}(iI_2, i, 1)$ to obtain the family

$$\tilde{f}^\lambda(u, v) = [f_1, f_2, f_3, -if_4]^t.$$

This is real for real values of λ . At $\lambda = 1$ it agrees with f^λ , being the same sphere embedded in the plane \mathbf{R}^3 . As λ varies over \mathbf{R}^* , the immersion moves through a family of isometrically embedded spheres, with constant curvature $c_\lambda \in (0, 1)$, in the de Sitter space S_1^3 .

Remark 3.5. Example 3.4 is not typical. As the parameter λ varies, one should not normally expect an embedding to remain an embedding, but, rather, only an immersion.

3.3. Relations between the immersions from different ranges of the spectral parameter. The deformation parameter will not generally appear in the maps f^λ in such a simple manner as occurred in Example 3.4. However, at the level of the Maurer-Cartan form, it is always the same, and therefore one has the following global result concerning the maps f^λ , \hat{f}^λ and \tilde{f}^λ in general:

Proposition 3.1. *Let $f_j^{\lambda_j}$, $j = 1, \dots, 2$, be two maps from any one of the Tables 2-5, obtained from the $(m+1)$ 'th column of F or $\text{Ad}_T F$, as described in Theorem 3.2, evaluated at two given points $\lambda_j \in (i\mathbf{R}^* \cup \mathbf{R}^* \cup S^1) \setminus \{\pm i\}$. Then:*

- (1) *If $f_1^{\lambda_1}$ is immersive at a point $x \in M$, then so is $f_2^{\lambda_2}$.*
- (2) *If $f_1^{\lambda_1}$ is a complete immersion then so is $f_2^{\lambda_2}$.*

Proof. The matrix T used to go between f_1 and f_2 is among the following list: I , $\text{diag}(iI_m, 1, I_k)$, $\text{diag}(iI_m, 1, iI_k)$, $\text{diag}(iI_m, i, I_k)$, $\text{diag}(iI_m, i, iI_k)$ and $\text{diag}(I_m, 1, iI_k)$, together with their compositions and inverses. It follows that the Maurer-Cartan forms of F and $\text{Ad}_T F$ are both of the form:

$$\begin{bmatrix} \omega & \\ \pm(i)^r(\lambda + \lambda^{-1})\theta & \pm(i)^s(\lambda - \lambda^{-1})\beta \end{bmatrix}^t \begin{bmatrix} (i)^r(\lambda + \lambda^{-1})\theta & (i)^s(\lambda - \lambda^{-1})\beta \\ & \eta \end{bmatrix},$$

for some fixed real matrix valued 1-forms ω , θ , β and η . Thus the coframe for $f_j^{\lambda_j}$, obtained from the first m components of the $(m+1)$ 'th column, is just a non-zero constant k_j times the fixed column vector valued 1-form θ . The relevant reality condition ensures that this constant is real. The condition for $f_j^{\lambda_j}$ to be an immersion is that the coframe consist of m linearly independent 1-forms, which proves the first part of the proposition.

For completeness, if $f_j^{\lambda_j}$ is an immersion with coframe $k_j\theta = k_j[\theta_1, \dots, \theta_m]^t$, then the induced metric is

$$k_j^2(\theta_1^2 + \dots + \theta_m^2).$$

Thus the induced metrics for the two immersions are positive constant multiples of each other, and hence completeness is equivalent for them. \square

Corollary 3.6. *Within any one of Tables 2-5 of Theorem 3.2, the existence problem for an isometric immersion with flat normal bundle $M_c^m \rightarrow N_{\tilde{c}}^{m+k}$, $c \neq \tilde{c}$, where M_c^m is a complete simply connected m -dimensional space form of constant curvature c in one of the appropriate intervals, and $N_{\tilde{c}}^{m+k}$ is the corresponding target space of constant curvature \tilde{c} , is equivalent throughout the table.*

Proof. This follows from the converse part of Theorem 3.2, together with Proposition 3.1. \square

3.4. Applications. An interesting application of Corollary 3.6 is to generalisations of the well known theorem of Hilbert that H^2 cannot be globally immersed into Euclidean space E^3 [10].

Corollary 3.7. *Let c be a negative real number. The problems of globally isometrically immersing the m -dimensional simply connected space form Q_c^m into the space forms $Q_{\tilde{c}}^{2m-1}$, for $c < \tilde{c}$, with $0 \neq \tilde{c}$, are all equivalent.*

Proof. For $c < \tilde{c}$, the normal bundle is automatically flat in this codimension [12]. Thus our problem is in the realm of Table 2 of Theorem 3.2. The case $\tilde{c} > 0$ belongs to the first row of Table 2, after rescaling the sphere so that $\tilde{c} = 1$. The case $\tilde{c} < 0$ fits into the third line of the table, after rescaling so that $\tilde{c} = -1$. Hence, Corollary 3.6 implies the result. \square

As mentioned in the introduction, Corollary 3.7 is a known result, but our proof does not depend on special coordinates.

Another application is:

Corollary 3.8. (1) *There is no global isometric immersion with flat normal bundle of a sphere $S^m(R)$, of dimension m and any radius R , into the pseudo-Riemannian hyperbolic space H_{m-1}^{2m-1} .*
 (2) *There is no global isometric immersion with flat normal bundle of a sphere $S^m(R)$, of dimension m and radius $R < 1$, into the pseudo-Riemannian sphere S_{m-1}^{2m-1} .*

Proof. These follow from JD Moore's proof [12] that a sphere $S^m(R)$ of radius $R > 1$, or equivalently Gauss curvature less than 1, cannot be globally immersed into S^{2m-1} . Together with Corollary 3.6, this says that complete immersions are not possible in Table 3 of Theorem 3.2, which accounts for both cases of this corollary. \square

Remark 3.9. In the special case that $m = 2$, then the normal bundle is flat, so Corollary 3.8 reproduces the result of Li [11], that there is no isometric immersion of a 2-sphere of constant curvature greater than 1 into the de Sitter space S_1^3 .

4. THE DPW METHOD

The next goal of this paper is to relate the loop group formulation of isometric immersions to that of pluriharmonic maps. For this we will need the generalised DPW method. The DPW method was first used in [6] to produce harmonic maps from a Riemann surface into a symmetric space from holomorphic data. It was extended to pluriharmonic maps in [5]. The main idea of the method was shown to be extendable to somewhat arbitrary connection order $\begin{pmatrix} b \\ a \end{pmatrix}$ maps in [2], to which we refer the reader for more details of the following sketch.

Let \mathcal{H} be a Birkhoff decomposable subgroup of ΛG , and $a \leq 0 \leq b$ be extended integers. The generalised DPW method gives the following bijection:

$$(13) \quad F \in \frac{\mathcal{H}}{\mathcal{H}^0}(M)_a^b \leftrightarrow \begin{aligned} F_+ &\in \mathcal{H}(M)_1^b, \\ F_- &\in \mathcal{H}(M)_a^{-1}, \end{aligned}$$

which holds provided elements to be factored are in the big cell of ΛG . The maps F_+ and F_- are simply the left factors in the left and right Birkhoff decompositions

$$(14) \quad F = F_+ G_- = F_- G_+.$$

The method was used in this form in [19] for pseudospherical surfaces in \mathbf{R}^3 , in which case the functions F_+ and F_- were functions of independent variables, simplifying the problem.

If τ is a negative involution of \mathcal{H} , and $F \in \frac{\mathcal{H}}{\mathcal{H}^0_\tau}(M)_a^b$, then it follows that $a = -b$ and, applying τ to (14), we deduce from uniqueness of the Birkhoff factorisation that we must have $F_- = \tau F_+$. In fact it requires some work to prove the \leftarrow side of the correspondence here, but what one has is

$$(15) \quad F \in \frac{\mathcal{H}_\tau}{\mathcal{H}^0_\tau}(M)_{-b}^b \leftrightarrow F_+ \in \mathcal{H}(M)_1^b.$$

For the main purpose of what follows, we only need to know that the bijection (15) always holds in a neighbourhood of the identity, although it is true that if τ is the S^1 reality condition (4), where τ^0 defines a compact real form of G , then the \leftarrow correspondence of (15) is global on M , as it is constructed from an Iwasawa splitting of the loop group, which holds globally.

5. CURVED FLATS

Curved flats were defined in [7] as follows: let U/K be a semisimple symmetric space defined by the commuting involutions σ^0 and ρ^0 of a complex semisimple Lie group G , where U is the fixed point set of the reality condition ρ^0 and K is the fixed point set of both involutions. Let M be a connected manifold of dimension m . The map $f : M \rightarrow U/K$ is a curved flat if $f^*R = 0$ as a 2-form on M , where R is the curvature tensor of U/K .

Here we are principally interested in the loop group formulation which defines a *family* of curved flats f^λ , parameterised by λ in the nonzero real numbers \mathbf{R}^* . To define these, extend ρ^0 to an involution ρ of ΛG by the formula (3), so that elements of ΛG_ρ are in U for $\lambda \in \mathbf{R}^*$. We also extend σ^0 to an involution of ΛG by the formula (2).

Let $\mathcal{H} := \Lambda G_\sigma$, the fixed point subgroup of σ . Note that ρ and σ commute, and we define $\mathcal{H}_\rho := \Lambda G_{\rho\sigma}$ to be the subgroup of elements of ΛG fixed by both involutions. It is shown in [7] that a (family of lifts into U of) curved flats is just an element of $\mathcal{H}_\rho(M)_0^1$, that is, a map F from M into \mathcal{H}_ρ whose Maurer-Cartan form has the expansion

$$A^\lambda := F_\lambda^{-1} dF_\lambda = A_0 + A_1 \lambda.$$

The involution σ enforces that A_0 and A_1 are in the $+1$ and -1 eigenspaces, \mathfrak{k} and \mathfrak{p} , respectively of σ^0 , and the *curved flat equations* for F , namely

$$(16) \quad \begin{aligned} dA_0 + A_0 \wedge A_0 &= 0, \\ dA_1 + A_0 \wedge A_1 + A_1 \wedge A_0 &= 0, \\ A_1 \wedge A_1 &= 0, \end{aligned}$$

are equivalent to the fact that A^λ satisfies the Maurer-Cartan equation $dA + A \wedge A = 0$ for all λ . This is the integrability condition for F , so it must hold.

In fact, since A_0 is in \mathfrak{k} , the Lie algebra of K , and is itself integrable by (16), we can gauge away this term by right multiplication by a map into K . In other words, the same family of maps into U/K is represented by a map \hat{F}_λ whose Maurer-Cartan form has the expansion

$$\hat{A}^\lambda = \hat{A}_1 \lambda.$$

We therefore make the following definition:

Definition 5.1. Let G , U , K , σ and ρ be as above. A (*normalised*) *extended curved flat* from M into U/K is an element of $\mathcal{H}_\rho(M)_1^1$.

6. PLURIHARMONIC MAPS INTO SYMMETRIC SPACES

Harmonic maps were first studied in the loop group setting by Uhlenbeck [20], and this was extended to pluriharmonic maps by Ohnita and Valli [14]. For more details on the formulation described here, as well as a discussion of methods to produce *finite type* examples, the reader could consult [3]. The geometrical interpretation of the spectral parameter deformation is described in [5]. We will proceed directly to the loop group formulation here.

Let \hat{U}/\hat{K} be a semisimple symmetric space given by the involution σ^0 and reality condition $\hat{\tau}^0$ of G . In a later section we will assume that G and σ^0 are those given in Section 5, while the reality conditions $\hat{\tau}^0$ and ρ^0 will be different.

We extend σ^0 to ΛG again by the formula (2), but this time we extend our reality condition $\hat{\tau}^0$ in a different way, by the rule (4), so that elements of $\Lambda G_{\hat{\tau}}$ are \hat{U} -valued for unitary values of λ .

Let $M_{\mathbf{C}}$ be a simply connected m dimensional complex manifold. As before, let $\mathcal{H} := \Lambda G_\sigma$. In [3] it is shown that an extended lift for a pluriharmonic map from $M_{\mathbf{C}}$ into \hat{U}/\hat{K} is given by an element F of $\mathcal{H}_{\hat{\tau}}(M_{\mathbf{C}})_1^1$, with one additional property, namely, that if one expands the Maurer-Cartan form of F ,

$$(17) \quad A^\lambda := F_\lambda^{-1} dF_\lambda = A_{-1} \lambda^{-1} + A_0 + A_1 \lambda^1,$$

and

$$A_i = A'_i + A''_i$$

is the decomposition of the 1-form A_i into its (1,0) and (0,1) components with respect to a complex basis for the cotangent space, then

$$A''_1 = 0.$$

Together with the fact that A^λ is fixed by $\hat{\tau}$ this also means that $A'_{-1} = 0$ and $A''_{-1} = \overline{A'_1}$, so that we have an expression

$$A^\lambda := \overline{A'_1} \lambda^{-1} + A_0 + A'_1 \lambda.$$

The extended pluriharmonic map into \hat{U}/\hat{K} associated to an extended lift $F \in \mathcal{H}_{\hat{\tau}}(M_{\mathbf{C}})_1^1$ is its equivalence class modulo right multiplication by a map into \hat{K} , which is just the subgroup $\mathcal{H}_{\hat{\tau}}^0$ of constant loops, hence we make the following definition:

Definition 6.1. Let G , σ , $\hat{\tau}$, \hat{U} and \hat{K} be as above. An *extended pluriharmonic map* from $M_{\mathbf{C}}$ into \hat{U}/\hat{K} is an element F of $\frac{\mathcal{H}_{\hat{\tau}}}{\mathcal{H}_{\hat{\tau}}^0}(M_{\mathbf{C}})_1^1$, with the property that if $F_\lambda^{-1} dF_\lambda$ has the expansion (17), then

$$(18) \quad A''_1 = 0.$$

We will denote the set of these by $\mathcal{P}_{\frac{\mathcal{H}_{\hat{\tau}}}{\mathcal{H}_{\hat{\tau}}^0}}(M_{\mathbf{C}})_1^1$

Remark 6.2. For every element $F \in \mathcal{P}_{\mathcal{H}_{\hat{\tau}}^0}(M_{\mathbf{C}})_1^1$ there is a unique pluriharmonic map $f : M_{\mathbf{C}} \rightarrow \widehat{U}/\widehat{K}$ obtained by evaluating F at $\lambda = 1$, and vice versa [5].

6.1. DPW for pluriharmonic maps. Now if one applies the splitting described in Section 4 to an extended pluriharmonic map as given in Definition 6.1 then it is straightforward to check that the condition (18) implies that the map F_+ on the right hand side of (15) is *holomorphic* in the $M_{\mathbf{C}}$ variables, that is $\bar{\partial}F_+ = 0$. Conversely, one can show that if one starts with an element $F_+ \in \mathcal{H}(M_{\mathbf{C}})_1^1$ which is holomorphic on $M_{\mathbf{C}}$, then the map F given by the left hand side of (15) is pluriharmonic. Now even though the DPW correspondence (15) only holds on the big cell in general, we can always renormalise our extended pluriharmonic map F at any point q by premultiplying it by $F^{-1}(q, \lambda)$. This is again a pluriharmonic map, and in the big cell on a neighbourhood of q , and therefore we have an alternative global characterisation of pluriharmonic maps. We summarise this as:

Proposition 6.1. *Let $G, \sigma, \hat{\tau}, \widehat{U}$ and \widehat{K} be given as in Definition 6.1. Suppose F is an element of $\frac{\mathcal{H}_{\hat{\tau}}}{\mathcal{H}_{\hat{\tau}}^0}(M_{\mathbf{C}})_1^1$. Then F is an extended pluriharmonic map if and only if the corresponding $F_+ \in \mathcal{H}(M_{\mathbf{C}})_1^1$ from the right hand side of (15) is holomorphic on $M_{\mathbf{C}}$.*

Remark 6.3. Note that this test is understood to be applied locally by renormalizing F . The holomorphic function F_+ is not in general defined globally.

7. PLURIHARMONIC MAPS CONSTRUCTED FROM ANALYTIC CURVED FLATS

Let $G, U, K, \sigma, \rho, \widehat{U}, \widehat{K}$ and $\hat{\tau}$ be as in Sections 5 and 6, and assume that the reality conditions ρ^0 and $\hat{\tau}^0$ commute, so that the extended involutions given by (3), (2) and (4) commute also. Let M be a connected paracompact real analytic manifold of dimension m . By taking an atlas of M and analytically extending the transition functions, we can embed M as a totally real submanifold of some complex manifold $M_{\mathbf{C}}$ of complex dimension m . The following theorem always holds at least locally on M , and globally if \widehat{K} is compact.

Theorem 7.1. *Let $f_+ : M \rightarrow U/K$ be a real analytic curved flat, represented by the extended family*

$$F_+ \in \mathcal{H}_{\rho}(M)_1^1.$$

Then there exists an open submanifold M_{ϵ} of $M_{\mathbf{C}}$, containing M , and a unique pluriharmonic map $\hat{f} : M_{\epsilon} \rightarrow \widehat{U}/\widehat{K}$, represented by

$$\hat{F} \in \mathcal{P}_{\frac{\mathcal{H}_{\hat{\tau}}}{\mathcal{H}_{\hat{\tau}}^0}}(M_{\epsilon})_{-1}^1,$$

such that the restriction of \hat{F} to M satisfies the reality condition ρ . More precisely we have the following correspondence from (15):

$$(19) \quad F = \hat{F}|_M \in \frac{\mathcal{H}_{\rho\hat{\tau}}}{\mathcal{H}_{\rho\hat{\tau}}^0}(M)_{-1}^1 \leftrightarrow F_+ \in \mathcal{H}_{\rho}(M)_1^1.$$

Proof. We need only to show that there is a holomorphic extension $\hat{F}_+ \in \mathcal{H}(M_{\epsilon})_1^1$ of F_+ , for some open M_{ϵ} containing M . Then the DPW correspondence (15) together with Proposition 6.1 gives us the required $\hat{F} \in \mathcal{P}_{\frac{\mathcal{H}_{\hat{\tau}}}{\mathcal{H}_{\hat{\tau}}^0}}(M_{\epsilon})_{-1}^1$. To see that \hat{F} restricts on M to an element of $\frac{\mathcal{H}_{\rho\hat{\tau}}}{\mathcal{H}_{\rho\hat{\tau}}^0}(M)_{-1}^1$, observe that $\hat{F}|_M$ is just the object obtained by applying the DPW correspondence (15) to F_+ itself, since this is done pointwise on M_{ϵ} . Since ρ is a positive involution, \mathcal{H}_{ρ} is Birkhoff decomposable, and so $\hat{F}|_M$ is fixed by ρ . The existence of \hat{F}_+ is shown in Proposition 7.1 below. \square

7.1. Complexifying real analytic curved flats. To complete the proof of Theorem 7.1, we need to show there is a holomorphic extension of F_+ .

Proposition 7.1. *Let F_+ be a real analytic element of $\mathcal{H}_\rho(M)_1^1$, as above. There exists an open submanifold M_ϵ of $M_{\mathbf{C}}$, containing M , such that F_+ has a unique holomorphic extension to an element $\hat{F}_+ \in \mathcal{H}(M_\epsilon)_1^1$.*

Proof. Consider the Maurer-Cartan form

$$A_+ := F_+^{-1} dF_+ = \eta \lambda,$$

where η is an analytic \mathfrak{p} -valued 1-form on M satisfying the curved flat equations:

$$(20) \quad d\eta = 0,$$

$$(21) \quad \eta \wedge \eta = 0,$$

which are equivalent to the Maurer-Cartan equation for A_+ . Let $\mathfrak{p}_{\mathbf{C}}$ denote the complexification of \mathfrak{p} .

Lemma 7.2. *There exists an open submanifold, M_ϵ , of $M_{\mathbf{C}}$, containing M , such that η has a unique analytic extension to a $\mathfrak{p}_{\mathbf{C}}$ -valued holomorphic 1-form $\eta^{\mathbf{C}}$ on M_ϵ , satisfying the curved flat equations (20) and (21).*

Let $z^j = x^j + iy^j$ be local coordinates on $M_{\mathbf{C}}$. Then η has the expression

$$\eta = \sum_j \eta_j dx^j,$$

where each \mathfrak{p} valued function η_j is analytic in x^1, \dots, x^m on M . By standard theory of power series, there is a neighbourhood M_ϵ of M in $M_{\mathbf{C}}$ to which each function η_j has a holomorphic extension, $\hat{\eta}_j$, which takes values in $\mathfrak{p}_{\mathbf{C}}$, and this extension is unique. Define $\hat{\eta} := \sum \hat{\eta}_k dz^k$. We assumed our manifold $M_{\mathbf{C}}$ was constructed from M by analytically extending the transition functions defining M , from which it follows that $\hat{\eta}$ is a well defined global holomorphic extension of η .

Let $d = \partial + \bar{\partial}$ be the usual decomposition of the d operator into holomorphic and antiholomorphic parts. Now $\bar{\partial}_k \hat{\eta}_j = 0$, so we have

$$d\hat{\eta} = \sum_{j < k} (\partial_j \hat{\eta}_k - \partial_k \hat{\eta}_j) dz^j \wedge dz^k,$$

and the term $(\partial_j \hat{\eta}_k - \partial_k \hat{\eta}_j)$ is just the analytic extension of $(\frac{\partial \eta_k}{\partial x^j} - \frac{\partial \eta_j}{\partial x^k})$, which vanishes by (20). The argument for (21) is analogous, and this proves the lemma.

Now denote by \hat{A}_+ the family of 1-forms $\hat{\eta} \lambda$. By Lemma 7.2, \hat{A}_+ satisfies the Maurer-Cartan equations for all λ , and therefore integrates to a map $\hat{F}_+ \in \mathcal{H}(\widetilde{M}_\epsilon)_1^1$, where \widetilde{M}_ϵ is the universal cover of M_ϵ . \hat{F}_+ is uniquely determined by the normalisation at some base point in $\pi^{-1}(p)$, where p is the base point of M , and π is the projection $\widetilde{M}_\epsilon \rightarrow M_\epsilon$. Since $\hat{\eta}$ is holomorphic on \widetilde{M}_ϵ , so is \hat{F}_+ .

Finally, we need to show that \hat{F}_+ descends to a well defined function on M_ϵ . Every point of M has an evenly covered neighbourhood in M_ϵ , so by shrinking M_ϵ if necessary to be the union of these neighbourhoods, we may assume that any point x of M_ϵ has an evenly covered simply connected neighbourhood U which intersects M . Let $\pi^{-1}U = \cup_k \tilde{U}_k$, a union of disjoint open sets with biholomorphisms $\pi_k : \tilde{U}_k \rightarrow U$. Now \hat{F}_+ restricted to $\pi^{-1}M$ does descend to a well defined function on M , namely the function F_+ which we started with, and its holomorphic extension to any of the biholomorphic neighbourhoods U_k is unique.

Hence $\hat{F}_+ \circ \pi_k^{-1}$ must be the same function for any k , in other words $\hat{F}_+ \circ \pi^{-1}$ is a well defined function on M_ϵ . \square

8. ISOMETRIC IMMERSIONS BETWEEN SPACE FORMS AS PLURIHARMONIC MAPS

To simplify the statement of results, we consider only the three cases when the target space is a sphere, but note that these contain two of the hyperbolic cases, and the other, Case 4 above, can be handled in the same way as Case 2.

Theorem 8.1. *Let M be a simply connected, paracompact real analytic manifold of dimension m , and with fixed base point p . We can assume that M is embedded as a totally real submanifold of some m -dimensional complex manifold $M_{\mathbf{C}}$. Let $f : M \rightarrow S^{m+k}$ be an immersion with flat normal bundle and induced constant sectional curvature c in one of the following unions of intervals:*

$$\begin{aligned} I_1 &= (-\infty, 0), \\ I_2 &= (0, 1), \\ I_3 &= (1, \infty), \end{aligned}$$

and f^λ the corresponding extended family obtained by the scheme in [8]. Then there exists an open submanifold M_ϵ of $M_{\mathbf{C}}$, containing M , and a unique extended family of pluriharmonic maps into a symmetric space, $\hat{f}^\lambda : M_\epsilon \rightarrow U_i/K_i$, such that the restriction $\hat{f}^\lambda|_M$ is f^λ . The symmetric spaces U_i/K_i corresponding to $c \in I_i$ are as follows:

$$\begin{aligned} U_1/K_1 &= \frac{SO(m+k, 1)}{SO(m) \times SO(k, 1)}, \\ U_2/K_2 &= \frac{SO(m+1, k)}{SO(m) \times SO(k, 1)}, \\ U_3/K_3 &= \frac{SO(m+k+1)}{SO(m) \times SO(k+1)}. \end{aligned}$$

Theorem 8.1 follows from Proposition 8.1 below and its analogues for Cases 1 and 3.

Let us consider Case 2 first, a family of isometric immersions $M \rightarrow S^{m+k+1}$ given by an element of

$$(22) \quad \frac{\mathcal{H}_{\mu\rho_2}(M)_-^1}{\mathcal{H}_{\mu\rho_2}^0(M)_-^1},$$

as defined in Section 3.1. Observe that (22) looks rather like a special case of the set on the left hand side of (19), except for the fact that the involution μ is not a reality condition (it is not conjugate linear). However, one can describe this loop group in another way: replace μ with the involution τ_2 defined by

$$\tau_2 X(\lambda) := \text{Ad}_Q(\overline{X(1/\bar{\lambda})}).$$

Then τ_2 is of the form (4) for the reality condition $\tau_2^0(X) := \text{Ad}_Q(\overline{X})$, and due to the reality condition ρ_2 , which says that $\overline{X(1/\bar{\lambda})} = X(1/\lambda)$ for $X \in \mathcal{H}_{\rho_2}$, we see that

$$\frac{\mathcal{H}_{\rho_2\tau_2}(M)_-^1}{\mathcal{H}_{\rho_2\tau_2}^0(M)_-^1} = \frac{\mathcal{H}_{\rho_2\mu}(M)_-^1}{\mathcal{H}_{\rho_2\mu}^0(M)_-^1}.$$

Definition 8.2. An extended isometric immersion $M \rightarrow S^{m+k}$ with induced constant sectional curvature $c \in (0, 1]$ is an element of $\frac{\mathcal{H}_{\rho_2\tau_2}(M)_-^1}{\mathcal{H}_{\rho_2\tau_2}^0(M)_-^1}$, where $G = SO(m+k+1, \mathbf{C})$ and ρ_2 , σ and τ_2 are as defined in this section.

It is now easy to prove the following result:

Proposition 8.1. *Let F be a real analytic element of $\frac{\mathcal{H}_{\rho_2\tau_2}}{\mathcal{H}_{\rho_2\tau_2}^0}(M)_{-1}^1$, as given in Definition 8.2, where M is a real analytic manifold. Then there exists a complex manifold M_ϵ of dimension m , containing M as a totally real submanifold, and a unique extended pluriharmonic map $\hat{F} \in \mathcal{P}_{\frac{\mathcal{H}_{\tau_2}}{\mathcal{H}_{\tau_2}^0}}(M_\epsilon)_{-1}^1$, such that $\hat{F}|_M = F$.*

Proof. The big cell $B\mathcal{H}$ is a neighbourhood of the identity, so if p is the point in M at which F is normalised, then there is a neighbourhood U_p of p such that $F(x)$ takes values in $B\mathcal{H}$ for all x in U_p . Since the normalisation point is relevant here, we will use the notation $\mathcal{H}(M, x)$ to denote maps into \mathcal{H} which are normalised at x . Now on U_p we can use the DPW correspondence (15) to associate to F a unique extended curved flat $F_+ \in \mathcal{H}_{\rho_2}(U_p, p)_1^1$. The map F_+ is analytic on U_p , because the Birkhoff splitting is analytic. Theorem 7.1 then gives the extension to $\hat{F} \in \mathcal{P}_{\frac{\mathcal{H}_{\tau_2}}{\mathcal{H}_{\tau_2}^0}}(U_p^\epsilon, p)_{-1}^1$, where $U_p \subset U_p^\epsilon$ and U_p^ϵ is open in $M_\mathbb{C}$.

For a point $q \in M \setminus U_p$, we consider instead the map

$$R(x, \lambda) := F(q, \lambda)^{-1}F(x, \lambda).$$

Now $R^{-1}dR = F^{-1}dF$, so R is also an element of $\frac{\mathcal{H}_{\rho_2\tau_2}}{\mathcal{H}_{\rho_2\tau_2}^0}(M, q)_{-1}^1$, where here the normalisation is at q rather than p . We can therefore apply the same argument to extend R to $\hat{R} \in \mathcal{P}_{\frac{\mathcal{H}_{\tau_2}}{\mathcal{H}_{\tau_2}^0}}(U_q^\epsilon, q)_{-1}^1$, for some neighbourhood U_q^ϵ of q in $M_\mathbb{C}$. On U_q^ϵ we then define

$$\hat{F}_q(z, \lambda) := F(q, \lambda)\hat{R}(z, \lambda).$$

Now \hat{F}_q and \hat{R} have the same Maurer-Cartan form, and therefore, since a map being pluriharmonic is characterised by its Maurer-Cartan form, \hat{F}_q is pluriharmonic. In fact \hat{F}_q is an element of $\mathcal{P}_{\frac{\mathcal{H}_{\tau_2}}{\mathcal{H}_{\tau_2}^0}}(U_q^\epsilon, p)_{-1}^1$, where here the "normalisation" at p still makes sense for \hat{F}_q , even if p is not in U_q^ϵ , because \hat{F}_q clearly agrees with F on $M \cap U_q$, and therefore can be extended along M to p .

We now want to check that \hat{F}_q and \hat{F} agree at any point $w \in U_p^\epsilon \cap U_q^\epsilon$. Let r be a point in $U_p \cap U_q$. It is enough to show that

$$\hat{F}^{-1}(r, \lambda)\hat{F}_q(w, \lambda) = \hat{F}^{-1}(r, \lambda)\hat{F}(w, \lambda).$$

Now both the left and right hand side of this equation are normalised at r , because $r \in M$ and therefore $\hat{F}^{-1}(r, \lambda) = \hat{F}_q^{-1}(r, \lambda)$. In fact they are both elements of $\mathcal{P}_{\frac{\mathcal{H}_{\tau_2}}{\mathcal{H}_{\tau_2}^0}}(U_q^\epsilon \cap U_p^\epsilon, r)_{-1}^1$, which agree with $F^{-1}(r, \lambda)F(x, \lambda)$ for $x \in M \cap U_q^\epsilon \cap U_p^\epsilon$. Applying the DPW correspondence (15) to either one of them we get on the right hand side of (15) the holomorphic curved flat $\hat{H}_+ \in \mathcal{H}(U_q^\epsilon \cap U_p^\epsilon, r)_1^1$, which has to be the unique holomorphic extension of the curved flat corresponding to $F^{-1}(r, \lambda)F(x, \lambda)$. Hence, by the uniqueness of the correspondence (15), $\hat{F}^{-1}(r, \lambda)\hat{F}_q(w, \lambda)$ and $\hat{F}^{-1}(r, \lambda)\hat{F}(w, \lambda)$ must be equal.

Repeating the same procedure for any other point $s \in M \setminus \{U_p \cup U_q\}$, an identical argument also shows that \hat{F}_s agrees with both \hat{F} and \hat{F}_q on the intersections of their respective domains of definition, and thus, taking M_ϵ to be the union over all $x \in M$ of the sets U_x^ϵ we have the required global extension \hat{F} . \square

Proposition 8.1 says that a real analytic element of $\frac{\mathcal{H}_{\rho_2\tau_2}}{\mathcal{H}_{\rho_2\tau_2}^0}(M)_{-1}^1$, can be extended to a family of pluriharmonic maps into $G_{\tau_2^0}/G_{\tau_2^0\sigma^0}$. Let us identify the real form $G_{\tau_2^0}$:

Lemma 8.3. *The real form $G_{\tau_2^0}$ is isomorphic to $SO(m+1, k, \mathbf{R})$.*

Proof. Let $J := \text{diag}(I_{m+1}, -I_k)$, and take $SO(m+1, k, \mathbf{R})$ to be the set of matrices in $GL(m+k+1, \mathbf{R})$ which satisfy

$$X^t J X = J.$$

Let $T := \text{diag}(I_{m+1}, iI_k)$. We check that Ad_T takes $G_{\tau_2^0}$ isomorphically to $SO(m+1, k, \mathbf{R})$: if $X \in G_{\tau_2^0} = SO(m+k+1, \mathbf{C})_{\tau_2^0}$, then the $SO(m+k+1)$ condition $X^t X = I$ implies

$$(\text{Ad}_T X)^t J (\text{Ad}_T X) = J = T^{-1} X^t X T^{-1} = T^{-1} T^{-1} = J,$$

and the condition $\tau_2^0 X = \overline{\text{Ad}_Q X} = X$ implies

$$\overline{\text{Ad}_R X} = \overline{\text{Ad}_Q \text{Ad}_T \text{Ad}_Q X} = \text{Ad}_Q \text{Ad}_{\bar{T}} X = \text{Ad}_T X.$$

Thus $\text{Ad}_T X$ preserves J and is real. The converse is similar. \square

Now G_{σ^0} is the subgroup of matrices made up of diagonal blocks which are $m \times m$ and $(k+1) \times (k+1)$ respectively, and one sees that

$$G_{\tau_2^0 \sigma^0} = SO(m, \mathbf{R}) \times SO(k, 1, \mathbf{R}),$$

which completes the proof of the first part of Theorem 8.1.

The other two cases can be treated similarly:

Case 1

These are given by elements of

$$\frac{\mathcal{H}_{\rho_1 \mu}}{\mathcal{H}_{\rho_1 \mu}^0}(M)_{-1}^1.$$

Since ρ_1 , given by

$$\rho_1 X(\lambda) := \overline{X(-\bar{\lambda})},$$

is a reality condition along $i\mathbf{R}$ rather than \mathbf{R} we cannot define τ as we did in Case 2. Instead we define

$$\tau_1 X(\lambda) := \text{Ad}_Q \text{Ad}_P \overline{X(1/\bar{\lambda})},$$

where we recall that $(\sigma X)(\lambda) := \text{Ad}_P X(-\lambda)$, and $(\mu X)(\lambda) := \text{Ad}_Q(X(1/\lambda))$.

Again, τ_1 is an involution of the type (4) for the reality condition

$$\tau_1^0 X = \text{Ad}_Q \text{Ad}_P \overline{X},$$

and, for $X \in \mathcal{H}_{\rho_1}$ we have

$$\begin{aligned} \tau_1(X(\lambda)) &= \text{Ad}_Q \sigma \rho_1 X(1/\lambda) \\ &= \text{Ad}_Q X(1/\lambda) \\ &= \mu X(\lambda). \end{aligned}$$

From this it follows that

$$\mathcal{H}_{\rho_1 \tau_1} = \mathcal{H}_{\rho_1 \mu},$$

and we can proceed as in Case 2. Note that, although ρ_1 is not a reality condition along \mathbf{R} , as ρ was, the only thing that mattered in our construction of pluriharmonic maps from curved flats was that ρ should be a positive reality condition, and this is the case for ρ_1 .

Evidently, elements of $\frac{\mathcal{H}_{\rho_1 \mu}}{\mathcal{H}_{\rho_1 \mu}^0}(M)_{-1}^1$ extend to pluriharmonic maps into $G_{\tau_1^0}/G_{\tau_1^0 \sigma^0}$. Now $\text{Ad}_Q \text{Ad}_P = \text{Ad}_{QP}$, where $QP = \text{diag}(I_{m+k}, -1)$, and so the same argument given in Lemma 8.3 shows that the real form $G_{\tau_1^0}$ is isomorphic to $SO(m+k, 1, \mathbf{R})$, while the subgroup $G_{\tau_1^0 \sigma^0}$ is $SO(m, \mathbf{R}) \times SO(k, 1, \mathbf{R})$.

Case 3

Here the relevant loop group is $\mathcal{H}_{\rho_3\mu}$, and ρ_3 is the negative reality condition:

$$(\rho_3 X)(\lambda) := \overline{X(1/\bar{\lambda})}.$$

In this case we define a positive involution $\hat{\rho}_3$ by

$$\begin{aligned} \hat{\rho}_3 X(\lambda) &:= \rho_3 \mu X(\lambda) \\ &= \text{Ad}_Q \overline{X(\bar{\lambda})}, \end{aligned}$$

and one easily verifies that

$$\mathcal{H}_{\hat{\rho}_3\mu} = \mathcal{H}_{\rho_3\mu}.$$

This is then the same as the first case, as $\hat{\rho}_3$ is an involution of the form (3) for the reality condition

$$\hat{\rho}_3^0 X = \text{Ad}_Q \overline{X}.$$

Proceeding as in Case 2, we replace μ with τ_3 defined by:

$$\begin{aligned} (\tau_3 X)(\lambda) &:= (\mu \hat{\rho}_3 X)(\lambda) \\ &= \overline{X(1/\bar{\lambda})}. \end{aligned}$$

Thus an element of $\mathcal{H}_{\tau_3}(M_\epsilon)_{-1}^1$ is a pluriharmonic map into

$$U/K = SO(m+k+1, \mathbf{R}) / (SO(m, \mathbf{R}) \times SO(k+1, \mathbf{R})),$$

and this completes the proof of Theorem 8.1.

8.1. Some remarks on Theorem 8.1.

- (1) A similar result holds even if M is not simply connected, the only difference being that the associated family f^λ is in general only defined on the universal cover.
- (2) The original map f is actually a map into the last of these symmetric spaces, U_3/K_3 , which is not isomorphic to the first two, and so the result might seem odd, but the explanation is that the pluriharmonic maps are obtained by evaluating the extended map at $\lambda \in S^1$, while in the first two cases, the isometric immersions are obtained from values of λ in $i\mathbf{R}^*$ and \mathbf{R}^* respectively. On the intersections, $\lambda = \pm 1$ and $\lambda = \pm i$, the maps take their values in the intersections of the relevant symmetric spaces.

8.2. Totally geodesic immersions. Recall from Section 3 that totally geodesic immersions into the sphere and hyperbolic space were obtained from the extended families of immersions evaluated at $\lambda = \pm 1$, but that given a totally geodesic immersion f into the sphere, there is no canonical way to insert the parameter λ to obtain the extended family. Nor is there an obvious way to tell whether such a family exists for f (in contrast to other values of the induced curvature c). To see that the question is meaningful, consider a totally geodesic embedding of S^n into S^{2n-1} . This cannot belong to one of the families of Case 2 above, that is with constant curvature varying in the interval $(0, 1]$, because the immersion condition is preserved within these families, and, as noted before, there is no global isometric immersion of a sphere S_c^n , of radius $1/\sqrt{c}$ for $0 < c < 1$, into S^{2n-1} [12].

As an application of Theorem 8.1, we characterise in terms of pluriharmonic maps, which real analytic totally geodesic submanifolds of S^{m+k} can be extended to one of the families discussed here. The analogue holds for totally geodesic submanifolds of H^{m+k} . The point here is that, while there is no canonical way to insert the parameter λ into a totally geodesic immersion, there *is* for a pluriharmonic map.

Let M be as in Theorem 8.1, suppose $f : M \rightarrow S^{m+k}$ is a totally geodesic immersion with flat normal bundle, and regard f as a map into $SO(m+k+1)/SO(m) \times SO(k)$, given by its equivalence class of adapted frames. Since the second fundamental form of f is zero, one may check that f satisfies both the reality conditions τ_2^0 and τ_3^0 , which, combined with σ^0 define the symmetric spaces $SO(m+1, k)/(SO(m) \times SO(k, 1))$ and $SO(m+k+1)/(SO(m) \times SO(k+1))$ respectively. Around a point $p \in M \subset M_{\mathbf{C}}$, the real analytic function f can be extended locally to a pluriharmonic map from $M_{\mathbf{C}}$ into either of these symmetric spaces, since such maps are locally just the real parts of holomorphic functions. These extensions are not unique, however, since the notion of real part depends on your choice of coordinates for the target space.

The following is an immediate corollary of Theorem 8.1.

Proposition 8.2. *If $f : M \rightarrow S^{m+k}$ is a real analytic totally geodesic immersion with flat normal bundle, then:*

- (1) $f = f^1$ for some extended isometric immersion f^λ with $c_\lambda \in (0, 1]$ if and only if it has an extension to a pluriharmonic map $\hat{f} : M_{\mathbf{C}} \rightarrow SO(m+1, k)/(SO(m) \times SO(k, 1))$ such that, if \hat{f}^λ is the corresponding extended pluriharmonic map, then $\hat{f}^\lambda|_M$ is fixed by the involution ρ_2 .
- (2) $f = f^1$ for some extended isometric immersion f^λ with $c_\lambda \in [1, \infty)$ if and only if it has an extension to a pluriharmonic map $\hat{f} : M_{\mathbf{C}} \rightarrow SO(m+k+1)/(SO(m) \times SO(k+1))$ such that, if \hat{f}^λ is the corresponding extended pluriharmonic map, then $\hat{f}^\lambda|_M$ is fixed by the involution $\hat{\rho}_3$.

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KOBE UNIVERSITY, 1-1, ROKKODAI, NADA-KU, KOBE 657-8501, JAPAN

E-mail address: `brander@math.kobe-u.ac.jp`